

Another proof of $\zeta(2) = \frac{\pi^2}{6}$ using double integrals

Daniele Ritelli *

Abstract

Starting from the double integral

$$\int_0^\infty \int_0^\infty \frac{dx dy}{(1+y)(1+x^2y)}$$

we give another solution to the Basel Problem

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

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The celebrated Euler identity, known as the Basel Problem,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad (1)$$

has been proved in many different ways. In this note, we focus on the derivation of (1), taking advantage of the nice interplay between a double integral and a geometric series as has appeared in several articles on this subject, [1, 5, 3]. As Sir Michael Atiyah declared in an interview [8]:

Any good theorem should have several proofs, the more the better. For two reasons: usually, different proofs have different strengths and weaknesses, and they generalize in different directions: they are not just repetitions of each other

we find that there is always something worthy of attention in a new proof of a known result. Here, we provide a proof that uses a rational function with the lowest degree among the functions used in different proofs of the same kind.

The author who inaugurated this approach was Apostol, [1], inspired by Beukers' paper [2], where the double integral

$$\int_0^1 \int_0^1 \frac{dx dy}{1-xy} \quad (2)$$

was utilized to prove the irrationality of $\zeta(2)$. Apostol, instead, evaluated (2) in two ways. First, by expanding $1/(1-xy)$ into a geometric series and then, with a change of variables corresponding to the rotation of the coordinate axes through the angle $\pi/4$ radians. By equating the expression so obtained, the value of $\zeta(2)$ is found. It is worth noting that for the second evaluation, one has to compute the elaborate integrals

$$I_1 = \int_0^{1/\sqrt{2}} \frac{1}{\sqrt{2-u^2}} \arctan\left(\frac{u}{\sqrt{2-u^2}}\right) du, \text{ and}$$

$$I_2 = \int_{1/\sqrt{2}}^{\sqrt{2}} \frac{1}{\sqrt{2-u^2}} \arctan\left(\frac{\sqrt{2-u}}{\sqrt{2-u^2}}\right) du$$

*Dipartimento di Scienze Statistiche, Università di Bologna daniele.ritelli@unibo.it

using suitable trigonometric changes of variables. The evaluation of Beukers, Calabi and Kolk, [3], is similar. They expand $1/(1 - x^2y^2)$ into a geometric series to obtain the $\zeta(2)$ series, after which they introduce the two dimensional trigonometric changes of variables

$$x = \frac{\sin u}{\cos v}, \quad y = \frac{\sin v}{\cos u}$$

to evaluate the double integral. The proof of Hirschhorn [6] stems from the double inequality

$$2 \left(\arctan \frac{a}{1 + \sqrt{1 - a^2}} \right)^2 < \sum_{n=0}^{\infty} \frac{a^{4n+2}}{(2n+1)^2} < 2 (\arctan a)^2, \quad 0 < a < 1, \quad (3)$$

and a passage to the limit as $a \rightarrow 1$. The derivation of (3) is based on an integral inequality with regard again to the function $f(x, y) = 1/(1 - x^2y^2)$. To obtain (3), two integrals of $f(x, y)$ over two different regions of the plane are computed. All these approaches have in common the need to remove a singularity at the point $(1, 1)$ of the integrand. Our proof is inspired by [5], where another definite integral

$$\int_0^{+\infty} \int_0^1 \frac{x}{(1+x^2)(1+x^2y^2)} dx dy \quad (4)$$

is computed, first by integrating with respect to x and then with respect to y and vice versa. The same integral is considered in the probabilistic proof given in [7], where integral (4) comes from the product of two positive Cauchy random variables. The proof of [5], as well our proof, uses functions with no singularity in the domain of integration, so we can consider that in same sense these proofs are simpler. Moreover, our proof uses a lower degree rational function than the one used in [5].

Our starting point, as with most of the papers on this subject, is that (1) is equivalent to

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}. \quad (5)$$

In our proof we will show (5) starting from the double integral

$$\int_0^{\infty} \int_0^{\infty} \frac{dx dy}{(1+y)(1+x^2y)}. \quad (6)$$

If we integrate (6) first with respect to x and then to y , we find that:

$$\begin{aligned} \int_0^{\infty} \left(\frac{1}{1+y} \int_0^{\infty} \frac{dx}{1+x^2y} \right) dy &= \int_0^{\infty} \left(\frac{1}{1+y} \left[\frac{\arctan(\sqrt{y}x)}{\sqrt{y}} \right]_{x=0}^{x=\infty} \right) dy \\ &= \frac{\pi}{2} \int_0^{\infty} \frac{dy}{\sqrt{y}(1+y)} = \frac{\pi}{2} \int_0^{\infty} \frac{2u}{u(1+u^2)} du = \frac{\pi^2}{2} \end{aligned} \quad (7)$$

where we used the change of variable $y = u^2$ in the last step. Reversing the order of integration yields

$$\begin{aligned} \int_0^{\infty} \left(\int_0^{\infty} \frac{dy}{(1+y)(1+x^2y)} \right) dx &= \int_0^{\infty} \frac{1}{1-x^2} \left(\int_0^{\infty} \left(\frac{1}{1+y} - \frac{x^2}{1+x^2y} \right) dy \right) dx \\ &= \int_0^{\infty} \frac{1}{1-x^2} \ln \frac{1}{x^2} dx = 2 \int_0^{\infty} \frac{\ln x}{x^2-1} dx. \end{aligned} \quad (8)$$

Hence, equating (7) and (8) we get

$$\int_0^{\infty} \frac{\ln x}{x^2-1} dx = \frac{\pi^2}{4}. \quad (9)$$

Now split the integration domain in (9) between $[0, 1]$ and $[1, \infty)$ and change the variable $x = 1/u$ in the second integral, so that

$$\begin{aligned} \int_0^{\infty} \frac{\ln x}{x^2-1} dx &= \int_0^1 \frac{\ln x}{x^2-1} dx + \int_1^{\infty} \frac{\ln x}{x^2-1} dx \\ &= \int_0^1 \frac{\ln x}{x^2-1} dx + \int_0^1 \frac{\ln u}{u^2-1} du. \end{aligned} \quad (10)$$

From (9) and (10) we get

$$\int_0^1 \frac{\ln x}{x^2 - 1} dx = \frac{\pi^2}{8}. \quad (11)$$

Equation (5) now follows, expanding, as in [5], the denominator of the integrand on the left hand side of (11) into a geometric series and using the Monotone Convergence Theorem (see [4] pp. 95-96). Thus, we have:

$$\int_0^1 \frac{\ln x}{x^2 - 1} dx = \int_0^1 \frac{-\ln x}{1 - x^2} dx = \sum_{n=0}^{+\infty} \int_0^1 (-x^{2n} \ln x) dx. \quad (12)$$

Integrating by parts yields

$$\int_0^1 (-x^{2n} \ln x) dx = \left[-\frac{x^{2n+1}}{2n+1} \ln x \right]_0^1 + \int_0^1 \frac{x^{2n}}{2n+1} dx = \frac{1}{(2n+1)^2} \quad (13)$$

so that considering (13), we can write (12) as

$$\int_0^1 \frac{\ln x}{x^2 - 1} dx = \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} \quad (14)$$

and we are done equating (11) and (14).

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*Dipartimento di Statistica, Università di Bologna,
Viale Filopanti, 5 Bologna
daniele.ritelli@unibo.it*